Solution to Assignment 11

Section 9.1: no. 8, 9, 11, 13.

(8). Take $a_n = \dfrac{(-1)^n}{\sqrt{n}}$. You may use definition to show it converges, but later you can use the Alternating Test.

(9). For $\varepsilon > 0$, there is some n_0 such that

$$|\sum_{k=m}^{n} a_k| < \varepsilon/2 \ , \quad \forall m, n \geq n_0 \ .$$

But then

$$na_n = (n - n_0)a_n + n_0 a_n \leq a_{n_0} + \cdots + a_n + n_0 a_n < \varepsilon/2 + n_0 a_n \ .$$

As $\sum a_n$ converges implies $\lim_{n\to\infty} a_n = 0$, we can find some $n_1 \geq n_0$ such that $n_0 a_n < \varepsilon/2$ for all $n \geq n_1$. Putting things together, for $n \geq n_1$,

$$0 \leq na_n < \frac{\varepsilon}{2} + n_0 a_n < 2 \times \frac{\varepsilon}{2} = \varepsilon \ .$$

(11). The assumption implies that there is some α and n_0 such that $|n^2 a_n - \alpha| \leq 1$ for all $n \geq n_0$. Therefore,

$$\left|\sum_{k=m}^{n} a_k\right| \leq (|\alpha| + 1) \sum_{k=m}^{n} \frac{1}{k^2} \ , \quad n, m \geq n_0 \ .$$

As $\sum n^{-2} < \infty$, for $\varepsilon > 0$, there is some $n_1 \geq n_0$ such that $\sum_{k=m}^{n} k^{-2} < \varepsilon/(|\alpha| + 1)$, so $|\sum_{k=m}^{n} a_k| < \varepsilon$ for all $m, n \geq n_1$ too.

(13a).

$$\frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n}} = \frac{1}{(\sqrt{n+1} + \sqrt{n})\sqrt{n}} \geq \frac{1}{2(n+1)} \ .$$

As $\sum 1/(n+1)$ is divergent, this series is also divergent.

(13b).

$$\frac{\sqrt{n+1} - \sqrt{n}}{n} = \frac{1}{(\sqrt{n+1} + \sqrt{n})n} \leq \frac{1}{n^{3/2}} \ .$$

As $\sum n^{-3/2} < \infty$, this series is absolutely convergent.

Supplementary Exercise

You should use the new definition of exponential, logarithmic, cosine and sine functions in the following problems.

1. Establish the following properties of the exponential and log functions: For every $\alpha > 0$,

(a)

$$\lim_{x\to\infty} \frac{x^\alpha}{e^x} = 0 \ ,$$

(b)

$$\lim_{x\to-\infty} |x|^\alpha e^x = 0 \ ,$$

(c)

$$\lim_{x\to\infty} \frac{\log x}{|x|^\alpha} = 0 \ ,$$

(d)

$$\lim_{x\to 0} |x|^\alpha \log|x| = 0 \ .$$

Solution. (a). We can fix some k such that $x^\alpha \le x^k$ for all $x \ge 1$. It suffices to prove (a) by assuming $\alpha = k$. Using the expression

$$e^x = E(x) = \sum_{n=0}^\infty \frac{x^n}{n!} \ge \frac{x^{k+1}}{(k+1)!} \ , \quad x > 0 \ ,$$

we have

$$0 \le \frac{x^k}{e^x} \le \frac{(k+1)!}{x^{k+1}} x^k = \frac{(k+1)!}{x} \to 0 \ , \quad \text{as } x \to \infty \ ,$$

done.

(b). Follow from (a) after letting x be $-x$.

(c). Letting $y = \log x, x \ge 1$, $\dfrac{\log x}{|x|^\alpha}$ is turned into $\dfrac{y}{e^{\alpha y}}$, and the desired conclusion follows from (a).

(d). Let $y = 1/|x|$ and then use (c).

Note. This exercise is about the growth of the exponential and logarithmic functions compared to powers. EVERY math major should know it.

2. Establish the following properties of the cosine and sine functions:

(a)

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \ ,$$

(b)

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2} \ ,$$

Solution. After we have rigorously proved the derivative of the sine function is cosine and $\cos 0 = 1$ as well as the L'Hospital Rule, we can use them to get (a). Similarly we have (b). An alternative way is to apply the Taylor's Expansion Theorem.

Note. Again EVERY math major should know this.

3. Study the improper integrability of the following integrals:

(a)

$$\int_0^1 x^{-1/4} \log x \, dx,$$

(b)
$$\int_0^1 \frac{(1-\cos x)\log x}{x^3}\,dx\ ,$$

(c)
$$\int_0^\infty \frac{\sin x}{e^x-1}\,dx\ ,$$

(d) Optional.
$$\int_0^\infty \frac{\sin x}{x}\,dx\ .$$

Solution. (a). The integrand is unbounded near 0. Using $\lim_{x\to 0^+} x^\alpha \log x = 0$ for every positive α, we let $\alpha = 1/4$. For $\varepsilon = 1$, there is some δ such that $|x^{1/4}\log x| < 1$ for $x \in (0,\delta)$. Hence $x^{-1/4}\log x \le x^{-1/4}x^{-1/4} = x^{-1/2}$. Therefore,

$$\left|\int_a^c x^{-1/4}\log x\,dx\right| \le \int_a^c x^{-1/2}\,dx = 2c^{1/2} - 2a^{1/2} \to 0\ ,$$

as $a, c \to 0$. By Cauchy Criterion, the improper integral exists.

(b) $(1-\cos x)/x^2$ tends to $1/2$. We can fix some δ such that $1/4 \le (1-\cos x)/x^2$ for $x \in (0,\delta)$. Then
$$\frac{(1-\cos x)|\log x|}{x^3} \ge \frac{|\log x|}{4x} \ge \frac{1}{4x}, \quad x \in (0,\rho)\ ,$$
for some $\rho \le \delta$ (to make sure that $|\log x| \ge 1$). Therefore,

$$\int_a^\rho \frac{(1-\cos x)\log x}{x^3}\,dx \ge \int_a^\rho \frac{1}{4x}\,dx \to \infty,$$

as $a \to 0^+$. We conclude that the improper integral does not exist.

(c) We use $e^x > 1 + x$ for $x \ge 0$ to get $0 \le \sin x/(e^x-1) \le \sin x/x \le 1$. Hence

$$\int_a^{a'} \frac{\sin x}{e^x-1}\,dx \le a' - a \to 0,$$

as $a' - a \to 0$. It shows that
$$\int_0^1 \frac{\sin x}{e^x-1}\,dx$$
exists. On the other hand, use $e^x > 1 + x^2/2$ to get $|\sin x|/(e^x-1) \le 2/x^2$. Hence

$$\int_b^{b'} \frac{|\sin x|}{e^x-1}\,dx \le \int_b^{b'} \frac{2}{x^2}\,dx = 2\left(\frac{1}{b} - \frac{1}{b'}\right) \to 0\ ,$$

as $b, b' \to \infty$. It shows
$$\int_1^\infty \frac{\sin x}{e^x-1}\,dx$$
exists too. Hence this improper integral exists.

(d) Sketchy proof. No trouble at 0. We express the integral as

$$\sum_{n=0}^\infty \int_{n\pi}^{(n+1)\pi} \frac{\sin x}{x}\,dx = \sum_{n=0}^\infty \int_0^\pi \frac{\sin(n\pi + y)}{n\pi + y}\,dy = \int_0^\pi \frac{(-1)^n \sin y}{n\pi + y}\,dy\ .$$

For two consecutive terms $2n$ and $2n+1$, they have different signs and

$$\int_0^\pi \frac{\sin y}{2n\pi + y}\, dy - \int_0^\pi \frac{\sin y}{(2n+1)\pi + y}\, dy = \int_0^\pi \frac{\pi \sin y}{(2n\pi + y)((2n+1)\pi + y)}\, dy \leq \frac{C}{n^2} ,$$

which shows that the improper integral exists.

4. Optional. Consider $\sum_{n=1}^\infty a_n$ and let $\sum_{n=1}^\infty b_n$ and $\sum_{n=1}^\infty c_n$ where $b_n = a_n^+$ and $c_n = a_n^-$ (so $a_n = a_n^+ - a_n^-$). Show that $\sum_{n=1}^\infty b_n$ and $\sum_{n=1}^\infty c_n$ both are divergent to infinity when $\sum_{n=1}^\infty a_n$ is conditionally convergent.

   Solution. In case one the these series is convergent, say $\sum b_n$, let us show that $\sum c_n$ is also convergent, so $\sum |a_n| = \sum b_n + \sum c_n$ is also convergent, contradicting that $\sum a_n$ is only conditionally convergent. Let $\varepsilon > 0$, there is some n_0 such that $|a_{m+1} + \cdots + a_n| < \varepsilon/2$ for all $n, m \geq n_0$. On the other hand, choose $n_1 \geq n_0$, $b_{m+1} + \cdots + b_n < \varepsilon/2$ for all $n, m \geq n_1$. By subtracting these two inequalities and by choosing indices properly, we have $c_{m+1} + \cdots + c_n < \varepsilon$ for all $n, m \geq n_1$, done.

5. Optional. Show that every conditionally convergent series admits a rearrangement which is divergent to infinity.

   Solution. Adapting the notation in the previous problem, first we pick $b_1, \cdots, b_{n_1}$ such that $b_1 + \cdots + b_{n_1} > 1 + c_1$. Next, add $-c_1$ to the finite sequence to get $\{b_1, b_2, \cdots, b_{n_1}, -c_1\}$. Then add $b_{n_1+1}, \cdots, b_{n_2}$ so that $b_1 + b_2 + \cdots + b_{n_1} - c_1 + b_{n_1+1} + \cdots + b_{n_2} > 2 + c_2$. Add $-c_2$ to get $\{b_1, b_2, \cdots, b_{n_1}, -c_1, b_{n_1+1}, \cdots, b_{n_2}, -c_2\}$. Then add $b_{n_2+1}, \cdots, b_{n_3}$ so that $b_1 + \cdots - c_2 + b_{n_2+1} + \cdots + b_{n_3} > 3 + c_3$. By repeating the construction, we obtain a rearrangement whose partial sum is greater than any n. Note that this is possible because $\sum b_n = \infty$.

   Note. A theorem of Riemann states that given any number s including $\pm\infty$, there is a rearrangement on a conditionally convergent series converging to this number. You may google for it.